ON THE CONSTRUCTION OF PERIODIC SOLUTIONS OF A NONAUTONOMOUS QUASILINEAR SYSTEM WITH ONE DEGREE OF FREEDOM NEAR RESONANCE IN THE CASE OF DOUBLE ROOTS OF THE EQUATION OF FUNDAMENTAL AMPLITUDES

(O POSTROENII PERIODICHESKIKH RESHENII NEAVTONOMNOI KVAZILINEINOI SISTEMY S ODNOI STEPEN'IU SVOBODY VBLIZI REZONANSA V SLUCHAE DVUKRATNYKH KORNEI URAVNENII OSNOVNYKH AMPLITUD)

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G.V. PLOTNIKOVA
(Moscow)

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In [1,2] the problem was investigated of the construction of periodic solutions of nonautonomous quasilinear systems with one degree of freedom for the case of simple roots of the equation of the fundamental amplitudes, and also for the case of multiple roots with some auxiliary conditions. It was shown that the solution was representable in these cases in the form of an infinite series involving only integer powers of a small parameter μ. In the present paper the problem is considered of the construction of periodic solutions of such systems for the case of double roots of the equation of the fundamental amplitudes in the form of power series with not only integer powers of μ but also with powers of $\mu^{1/2}$.

1. We shall consider the following oscillatory system with one degree of freedom:

$$\frac{d^2x}{dt^2} + m^2x = f(t) + \mu F\left(t, x, \frac{dx}{dt}, \mu\right) \tag{1.1}$$

The quantity μ is a small positive parameter. Let us assume that the function $f(t)$ is a continuous periodic function of time t with period 2π whose Fourier expansion does not contain harmonics of the mth order (m is an integer); the function F is assumed to be analytic in the variables x, x and μ, and to be periodic in t of period 2π.

The generated equation (when $\mu = 0$) has the general form

$$x_0(t) = \varphi(t) + A_0 \cos mt + \frac{B_0}{m} \sin mt$$

which depends on two arbitrary constants A_0 and B_0. The function $\varphi(t)$ is a particular periodic solution of (1.1) when $\mu = 0$.

We shall use the method of Poincaré for finding the periodic solution of the System (1.1) which reduces to the solution $x_0(t)$ when $\mu = 0$. Hereby we shall make use of the following initial conditions

$$x(0) = x_0(0) + \beta_1, \qquad \dot{x}(0) = \dot{x}_0(0) + \beta_2$$

where β_1 and β_2 are functions of μ which vanish when $\mu = 0$. According to [2], the solution $x(t, \beta_1, \beta_2, \mu)$ can be expressed in the form

$$x(t, \beta_1, \beta_2, \mu) = \varphi(t) + (A_0 + \beta_1) \cos mt + \frac{B_0 + \beta_2}{m} \sin mt +$$
$$+ \sum_{n=1}^{\infty} \left[C_n(t) + \frac{\partial C_n(t)}{\partial A_0}\beta_1 + \frac{\partial C_n(t)}{\partial B_0}\beta_2 + \frac{1}{2}\frac{\partial^2 C_n(t)}{\partial A_0^2}\beta_1^2 + \frac{\partial^2 C_n(t)}{\partial A_0 \partial B_0}\beta_1\beta_2 + \right.$$
$$\left. + \frac{1}{2}\frac{\partial^2 C_n(t)}{\partial B_0^2}\beta_2^2 + \ldots \right] \mu^n \tag{1.2}$$

Here

$$C_n(t) = \frac{1}{m}\int_0^t H_n(t_1) \sin m(t - t_1)\, dt_1 \tag{1.3}$$

The functions $H_n(t_1)$ are computed with the aid of the definite relations (1.10) to (1.12) of [2]. Let us write down the conditions for the periodicity of the function $x(t, \beta_1, \beta_2, \mu)$

$$\sum_{n=1}^{\infty} \left[C_n(2\pi) + \frac{\partial C_n}{\partial A_0}\beta_1 + \frac{\partial C_n}{\partial B_0}\beta_2 + \frac{1}{2}\frac{\partial^2 C_n}{\partial A_0^2}\beta_1^2 + \frac{\partial^2 C_n}{\partial A_0 \partial B_0}\beta_1\beta_2 + \frac{1}{2}\frac{\partial^2 C_n}{\partial B_0^2}\beta_2^2 + \ldots \right] \mu^n = 0$$
$$\sum_{n=0}^{\infty} \left[\dot{C}_n(2\pi) + \frac{\partial \dot{C}_n}{\partial A_0}\beta_1 + \frac{\partial \dot{C}_n}{\partial B_0}\beta_2 + \frac{1}{2}\frac{\partial^2 \dot{C}_n}{\partial A_0^2}\beta_1^2 + \frac{\partial^2 \dot{C}_n}{\partial A_0 \partial B_0}\beta_1\beta_2 + \frac{1}{2}\frac{\partial^2 \dot{C}_n}{\partial B_0^2}\beta_2^2 + \ldots \right] \mu^n = 0 \tag{1.4}$$

Here the derivatives C_n and C_n with respect to A_0 and B_0 are taken with $t = 2\pi$, $\beta_1 = \beta_2 = \mu = 0$. The left-hand sides of (1.4) are holomorphic functions in the neighborhood of $\beta_1 = \beta_2 = \mu = 0$. Equations (1.4) can be used to determine the values $\beta_1(\mu)$ and $\beta_2(\mu)$. Knowing these values, and also $C_n(t)$ in (1.3), one can find the periodic solution of (1.1) on the basis of (1.3).

In this manner, it is seen that our problem has been reduced to the determination of the conditions for the existence and construction of two implicit functions β_1 and β_2 of the variable μ [3].

For the determination of the constants A_0 and B_0 in the solution we have the equation of the fundamental amplitudes

$$C_1(2\pi) = 0, \qquad \dot{C}_1(2\pi) = 0 \tag{1.5}$$

The case when these equations reduce to identities will be considered separately in this paper in Section 3. We shall use the notation

$$\Delta = \begin{vmatrix} \partial C_1/\partial A_0 & \partial C_1/\partial B_0 \\ \partial \dot{C}_1/\partial A_0 & \partial \dot{C}_1/\partial B_0 \end{vmatrix}, \qquad \Delta_1 = \begin{vmatrix} \partial C_1/\partial B_0 & C_2 \\ \partial \dot{C}_1/\partial B_0 & \dot{C}_2 \end{vmatrix}$$

In the case of simple roots of Equation (1.5), the determinant $\Delta \neq 0$ and the functions $\beta_1(\mu)$, $\beta_2(\mu)$, $x(t, \beta_1, \beta_2, \mu)$ are constructed in the form of series involving only integer power of μ in the manner shown in [2].

2. The condition $\Delta = 0$ indicates that the roots of the equation of the fundamental amplitudes are not simple. In the sequel we shall consider only the case of double roots of Equation (1.5). It is easy to show that in this case

$$\Delta^* = \begin{vmatrix} \dfrac{\partial C_1}{\partial B_0} & \dfrac{\partial^2 \dot{C}_1}{\partial A_0^2}\left(\dfrac{\partial C_1}{\partial B_0}\right)^2 - 2\dfrac{\partial^2 \dot{C}_1}{\partial A_0 \partial B_0}\dfrac{\partial C_1}{\partial A_0}\dfrac{\partial C_1}{\partial B_0} + \dfrac{\partial^2 \dot{C}_1}{\partial B_0^2}\left(\dfrac{\partial C_1}{\partial A_0}\right)^2 \\ \dfrac{\partial \dot{C}_1}{\partial B_0} & \dfrac{\partial^2 C}{\partial A_0^2}\left(\dfrac{\partial \dot{C}_1}{\partial B_0}\right)^2 - 2\dfrac{\partial^2 C_1}{\partial A_0 \partial B_0}\dfrac{\partial \dot{C}_1}{\partial A_0}\dfrac{\partial \dot{C}_1}{\partial B_0} + \dfrac{\partial^2 C_1}{\partial B_0^2}\left(\dfrac{\partial \dot{C}_1}{\partial A_0}\right)^2 \end{vmatrix} \neq 0 \tag{2.1}$$

The expression Δ^* is not unique, for, since $\Delta = 0$, it is obvious that one may replace the differentiation with respect to A_0 by the differentiation with respect to B_0 in (2.1), and conversely. However, in the case when $C_1 = C_1(A_0)$, $C_1 = C_1(B_0)$ or $C_1 = C_1(B_0)$, $C_1 = C_1(A_0)$, the condition for double roots in (2.1) is not valid, and one must replace it by

$$\frac{\partial^2 C_1}{\partial A_0^2} \neq 0, \qquad \frac{\partial^2 \dot{C}_1}{\partial B_0^2} \neq 0 \quad \text{or} \quad \frac{\partial^2 C_1}{\partial B_0^2} \neq 0, \qquad \frac{\partial^2 \dot{C}_1}{\partial A_0^2} \neq 0$$

From the expression (2.1) for Δ^* it follows that not a single term $\partial C_1/\partial A_0$, $\partial C_1/\partial B_0$, $\partial C_1/\partial A_0$, or $\partial C_1/\partial B_0$ is zero. In view of the fact that $\partial C_1/\partial B_0 \neq 0$, we have, by the implicit function theory [3], the equation

$$\beta_2 = \gamma_0(\mu) + \gamma_1(\mu)\beta_1 + \gamma_2(\mu)\beta_1^2 + \ldots \tag{2.2}$$

as a direct consequence of the first equation of (1.4).

Here β_2 is a holomorphic function in the neighborhood of $\mu = \beta_1 = 0$; $\gamma_j(\mu)$ is holomorphic in the neighborhood of $\mu = 0$. We note that $\gamma_0(0) = 0$

because β_2 has to vanish when $\mu = 0$. In this case the functions $\gamma_j(\mu)$ are representable in the form

$$\gamma_j(\mu) = \sum_{i=0}^{\infty} M_{ij}\mu^i \qquad (j = 0, 1, 2, \ldots) \tag{2.3}$$

The constants M_{ij} can be determined by substituting (2.2), with the use of (2.3), in the first of Equations (1.4), and by equating to zero the coefficients of equal powers of μ and β_1. Thus one obtains the next set of equations

$$M_{10} = -C_2\left(\frac{\partial C_1}{\partial B_0}\right)^{-1}, \qquad M_{01} = -\frac{\partial C_1}{\partial A_0}\left(\frac{\partial C_1}{\partial B_0}\right)^{-1}, \qquad M_{02} = -\left(\frac{\partial C_1}{\partial B_0}\right)^{-3} P_1^2 \tag{2.4}$$

$$M_{20} = -\left(\frac{\partial C_1}{\partial B_0}\right)^{-3}\left[\frac{\partial C_1}{\partial B_0}\left(C_3\frac{\partial C_1}{\partial B_0} - C_2\frac{\partial C_2}{\partial B_0}\right) + \frac{1}{2}C_2^2\frac{\partial^2 C_1}{\partial B_0^2}\right]$$

$$M_{11} = -\left(\frac{\partial C_1}{\partial B_0}\right)^{-3}\left[\frac{\partial C_1}{\partial B_0}\left(\frac{\partial C_1}{\partial B_0}\frac{\partial C_2}{\partial A_0} - \frac{\partial C_1}{\partial A_0}\frac{\partial C_2}{\partial B_0}\right) + C_2\left(\frac{\partial^2 C_1}{\partial B_0^2}\frac{\partial C_1}{\partial A_0} - \frac{\partial^2 C_1}{\partial A_0\partial B_0}\frac{\partial C_1}{\partial B_0}\right)\right]$$

$$M_{30} = \left(\frac{\partial C_1}{\partial B_0}\right)^{-5}\left[-\left(\frac{\partial C_1}{\partial B_0}\right)^3 M_{20}\left(\frac{\partial C_2}{\partial B_0}\frac{\partial C_1}{\partial B_0} - \frac{\partial^2 C_1}{\partial B_0^2}C_2\right) + \frac{1}{6}\frac{\partial^3 C_1}{\partial B_0^3}\frac{\partial C_1}{\partial B_0}C_2^3 - \right.$$

$$\left. - \frac{1}{2}\frac{\partial^2 C_2}{\partial B_0^2}C_2^2\left(\frac{\partial C_1}{\partial B_0}\right)^2 + \frac{\partial C_3}{\partial B_0}\left(\frac{\partial C_1}{\partial B_0}\right)^3 C_2 - C_4\left(\frac{\partial C_1}{\partial B_0}\right)^4\right]$$

$$M_{03} = \left(\frac{\partial C_1}{\partial B_0}\right)^{-5}\left[P_1^2\left(\frac{\partial^2 C_1}{\partial A_0\partial B_0}\frac{\partial C_1}{\partial B_0} - \frac{\partial^2 C_1}{\partial B_0^2}\frac{\partial C_1}{\partial A_0}\right) - P_1^3\frac{\partial C_1}{\partial B_0}\right] \qquad \text{etc.}$$

Here we have used the notation

$$P_n^k = P_n^k\left(\frac{\partial C_1}{\partial B_0}, -\frac{\partial C_1}{\partial A_0}\right) = \frac{1}{k!}\left(\frac{\partial}{\partial A_0}\frac{\partial C_1}{\partial B_0} + \frac{\partial}{\partial B_0}\left(-\frac{\partial C_1}{\partial A_0}\right)\right]^k C_n \qquad \begin{pmatrix}k = 1, 2, \ldots\\ n = 1, 2, \ldots\end{pmatrix}$$

$$\dot{P}_n^k = \dot{P}_n^k\left(\frac{\partial C_1}{\partial B_0}, -\frac{\partial C_1}{\partial A_0}\right) = \frac{1}{k!}\left[\frac{\partial}{\partial A_0}\frac{\partial C_1}{\partial B_0} + \frac{\partial}{\partial B_0}\left(-\frac{\partial C_1}{\partial A_0}\right)\right]^k \dot{C}_n$$

Hence, for β_2 we have

$$\beta_2 = M_{10}\mu + M_{20}\mu^2 + M_{01}\beta_1 + M_{02}\beta_1^2 + M_{11}\mu\beta_1 + \ldots \tag{2.5}$$

Substituting this Expression (2.5) for β_2 into the second equation of the system (1.4), we obtain

$$N_{10}\mu + N_{20}\mu^2 + N_{11}\mu\beta_1 + N_{02}\beta_1^2 + N_{30}\mu^3 + N_{21}\mu^2\beta_1 + N_{12}\mu\beta_1^2 + N_{03}\beta_1^3 + \ldots = 0 \tag{2.6}$$

It is obvious that the terms with β_1 must be absent from this equation because

$$N_{01} = -\left(\frac{\partial C_1}{\partial B_0}\right)^{-1}\Delta = 0$$

Taking into account (2.1), we find that

$$N_{02} = -\Delta^* \left[2 \left(\frac{\partial C_1}{\partial B_0} \right)^2 \frac{\partial \dot{C}_1}{\partial B_0} \right]^{-1} \neq 0$$

The remaining coefficients N_{ij} can be computed quite easily. Thus,

$$N_{10} = \Delta_1 \left(\frac{\partial C_1}{\partial B_0} \right)^{-1}$$

$$N_{20} = \left(\frac{\partial C_1}{\partial B_0} \right)^{-2} \left(\frac{\partial \dot{C}_1}{\partial B_0} \right)^{-1} \left\{ \frac{\partial \dot{C}_1}{\partial B_0} \left[\frac{1}{2} \frac{\partial \dot{C}_1}{\partial B_0^2} C_2^2 - \frac{\partial \dot{C}_2}{\partial B_0} \frac{\partial C_1}{\partial B_0} C_2 + \left(\frac{\partial C_1}{\partial B_0} \right)^2 \dot{C}_3 \right] - \right.$$
$$\left. - \frac{\partial C_1}{\partial B_0} \left[\frac{1}{2} \frac{\partial^2 C_1}{\partial B_0^2} \dot{C}_2^2 - \frac{\partial C_2}{\partial B_0} \frac{\partial \dot{C}_1}{\partial B_0} \dot{C}_2 + \left(\frac{\partial \dot{C}_1}{\partial B_0} \right)^2 C_3 \right] \right\}$$

$$N_{11} = \left(\frac{\partial C_1}{\partial B_0} \frac{\partial \dot{C}_1}{\partial B_0} \right)^{-1} \left[\dot{C}_2 \left(\frac{\partial^2 \dot{C}_1}{\partial B_0^2} \frac{\partial C_1}{\partial A_0} + \frac{\partial^2 C_1}{\partial A_0 \partial B_0} \frac{\partial \dot{C}_1}{\partial B_0} - \frac{\partial^2 C_1}{\partial B_0^2} \frac{\partial \dot{C}_1}{\partial A_0} - \frac{\partial^2 \dot{C}_1}{\partial A_0 \partial B_0} \frac{\partial C_1}{\partial B_0} \right) + \right.$$
$$\left. + \frac{\partial \dot{C}_1}{\partial B_0} \left(\frac{\partial \dot{C}_2}{\partial A_0} \frac{\partial C_1}{\partial B_0} + \frac{\partial C_2}{\partial B_0} \frac{\partial \dot{C}_1}{\partial A_0} - \frac{\partial \dot{C}_2}{\partial A_0} \frac{\partial C_1}{\partial A_0} - \frac{\partial C_2}{\partial A_0} \frac{\partial \dot{C}_1}{\partial B_0} \right) \right] \quad \text{etc.} \tag{2.7}$$

One must find the implicit function $\beta_1 = \beta_1(\mu)$ from (2.6). Setting $\mu = 0$ in (2.6), we obtain

$$N_{02}\beta_1^2 + N_{03}\beta_1^3 + \ldots = 0 \tag{2.8}$$

According to a theorem of Weierstrass on implicit functions [3, 4], the number of implicit functions $\beta_1(\mu)$ determined by Equations (2.6) is equal to the smallest exponent in the expansion (2.8) in powers of β_1, i.e. in the case of double roots of the equation of fundamental amplitudes ($N_{02} \neq 0$), it is equal to two. Both implicit functions can be expanded into convergent series either in integer powers of μ, or in powers of $\mu^{1/2}$

$$\beta_1 = \sum_{n=1}^{\infty} A_{n/l} \mu^{n/l} \qquad l = 1, 2 \tag{2.9}$$

where $A_{n/l}$ are constant coefficients. In this case the different expansions cannot exist simultaneously. We shall consider the more interesting cases.

1. If $\Delta_1 \neq 0$, i.e. $N_{10} \neq 0$, then the expansion for β_1 has the form (2.9) when $l = 2$. For the determination of the coefficients $A_{1/2}$ and A_1 we have the following equations

$$\begin{gathered} N_{02} A_{1/2}^2 + N_{10} = 0 \\ 2N_{02} A_{1/2} A_1 + N_{03} A_{1/2}^3 + N_{11} A_{1/2} = 0 \\ 2N_{02} A_{1/2} A_{3/2} + \ldots = 0 \end{gathered} \tag{2.10}$$

Taking into account the expressions for N_{02} and N_{10}, we obtain from the first equation of (2.10) two values for $A_{1/2}$

$$A_{1/2}^{(1)} = \sqrt{\frac{2}{\Delta^*}\frac{\partial C_1}{\partial B_0}\frac{\partial \dot{C}_1}{\partial B_0}}, \qquad A_{1/2}^{(2)} = -\sqrt{\frac{2}{\Delta^*}\frac{\partial C_1}{\partial B_0}\frac{\partial \dot{C}_1}{\partial B_0}}$$

We are interested only in real expansions. In order that the expansions may be real it is necessary and sufficient that

$$\frac{\Delta_1}{\Delta^*}\frac{\partial C_1}{\partial B_0}\frac{\partial C_1}{\partial B_0} > 0 \tag{2.11}$$

On the basis of (2.4) we obtain the series (2.5) in the form

$$\beta_2 = B_{1/2}^{(k)}\mu^{1/2} + B_1^{(k)}\mu + \ldots \qquad (k = 1, 2)$$

where

$$B_{1/2}^{(1)} = -\frac{\partial C_1}{\partial A_0}\left(\frac{\partial C_1}{\partial B_0}\right)^{-1} A_{1/2}^{(1)}, \qquad B_{1/2}^{(2)} = -\frac{\partial C_1}{\partial A_0}\left(\frac{\partial C_1}{\partial B_0}\right)^{-1} A_{1/2}^{(2)}$$

$$B_1^{(k)} = M_{10} + M_{01}A_1^{(k)} + M_{02}A_{1/2}^{(k)2} \qquad (k = 1, 2) \text{ etc}$$

2. If $\Delta_1 = 0$, that is if $N_{10} = 0$, we shall make the substitution [3]

$$\beta_1 = (v + a_k)\,\mu \qquad (k = 1, 2) \tag{2.12}$$

in order to obtain the conditions for the existence of the expansions. In (2.12) a_1 and a_2 denote the roots of the equation

$$S = N_{20} + N_{11}a + N_{02}a^2 = 0 \tag{2.13}$$

Substituting β_1 from (2.12) into (2.6), and dividing the resulting equation by μ^2, we obtain

$$\begin{aligned} &v\,(N_{11} + 2N_{02}a_k) + \mu\,(N_{30} + N_{21}a_k + N_{12}a_k^2 + N_{03}a_k^3) + \\ &+ N_{02}v^2 + \mu v\,(N_{21} + 2N_{12}a_k + 3N_{03}a_k^2) + \mu^2 L + \ldots = 0 \end{aligned} \tag{2.14}$$

All the terms which are not written down in this equation are of degree higher than two in μ. Only the following cases can occur.

a) The roots of Equation (2.13) are simple, $a_1 \neq a_2$. Then

$$\left(\frac{\partial S}{\partial a}\right)_{a=a_k} = N_{11} + 2N_{02}a_k \neq 0$$

In this case the coefficient of v in (2.14) is not equal to zero, i.e. there exists an expansion for v in integer powers of μ

$$v = \sum_{n=1}^{\infty} A_{n+1}\mu^n \tag{2.15}$$

where the A_{n+1} are constant coefficients determined by means of the system of equations:

$$A_2(N_{11}+2N_{02}a_k)+N_{30}+N_{21}a_k+N_{12}a_k^2+N_{03}a_k^3=0$$
$$A_3(N_{11}+2N_{02}a_k)+\ldots=0 \qquad \text{etc.}$$

This system, obviously, has a unique solution for a given a_k. On the basis of (2.12), (2.5) and (2.15) we obtain

$$\beta_1^{(k)}=\mu a_k+A_2^{(k)}\mu^2+\ldots \qquad (k=1,2)$$
$$\beta_2^{(k)}=\mu B_1^{(k)}+B_2^{(k)}\mu^2+\ldots \tag{2.16}$$

where

$$B_1^{(k)}=M_{10}+M_{01}a_k, \qquad B_2^{(k)}=M_{20}+M_{01}A_2^{(k)}+M_{02}a_k^2+2M_{02}a_kA_2^{(k)}+M_{11}a_k$$

Hence, if the roots of Equation (2.13) are real and simple, we have two expansions for $\beta_1(\mu)$ and $\beta_2(\mu)$ in the form of series involving only integer powers of μ.

This case was considered in [2].

b) Next, let the roots of Equation (2.13) be real and repeated, i.e. $a_1 = a_2 = a$. It is now obvious that

$$N_{11}+2N_{02}a=0, \qquad N_{11}^2-4N_{20}N_{02}=0 \tag{2.17}$$

Hence, the coefficient of v in Equation (2.14) vanishes.

Let us denote the coefficient of μ in (2.14) by K. Substituting the expression for a from (2.17) into K, we obtain

$$K=(8N_{02})^{-3}(8N_{02}^3N_{30}-4N_{21}N_{11}N_{02}^2+2N_{12}N_{11}^2N_{02}-N_{03}N_{11}^3)$$

Here

$$N_{30}=\frac{\partial\dot C_1}{\partial B_0}M_{30}+\frac{\partial^2\dot C_1}{\partial B_0^2}M_{10}M_{20}+\frac{\partial\dot C_2}{\partial B_0}M_{20}+\frac{1}{6}\frac{\partial^3\dot C_1}{\partial B_0^3}M_{10}^3+ \\ +\frac{1}{2}\frac{\partial^2\dot C_2}{\partial B_0^2}M_{10}^2+\frac{\partial\dot C_3}{\partial B_0}M_{10}+\dot C_4$$

$$N_{03}=\frac{\partial\dot C_1}{\partial B_0}M_{03}+\frac{\partial^2\dot C_1}{\partial A_0\,\partial B_0}M_{02}-\frac{\partial^2\dot C_1}{\partial B_0^2}M_{01}M_{02}+\frac{1}{6}\frac{\partial^3\dot C_1}{\partial A_0^3}+\frac{1}{2}\frac{\partial^3\dot C_1}{\partial A_0^2\partial B_0}M_{01}+ \\ +\frac{1}{2}\frac{\partial^3\dot C_1}{\partial A_0\,\partial B_0^2}M_{01}^2+\frac{1}{6}\frac{\partial^3\dot C_1}{\partial B_0^3}M_{01}^3$$

$$N_{21}=\frac{\partial\dot C_1}{\partial B_0}M_{21}+\frac{\partial^2\dot C_1}{\partial A_0\,\partial B_0}M_{20}+\frac{\partial^2\dot C_1}{\partial B_0^2}(M_{10}M_{11}+M_{01}M_{20})+ \\ +\frac{\partial\dot C_2}{\partial B_0}M_{11}+\frac{\partial^3\dot C_1}{\partial A_0\,\partial B_0^2}M_{10}^2+\frac{1}{2}\frac{\partial^3\dot C_1}{\partial B_0^3}(M_{10}^2M_{01})+\frac{\partial^2\dot C_2}{\partial A_0\,\partial B_0}M_{10}+$$

$$+\frac{\partial^2\dot{C}_2}{\partial B_0^2}M_{10}M_{01}+\frac{\partial\dot{C}_3}{\partial A_0}+\frac{\partial\dot{C}_3}{\partial B_0}M_{01}$$

$$N_{12}=\frac{\partial\dot{C}_1}{\partial B_0}M_{12}+\frac{\partial^2\dot{C}_1}{\partial A_0\,\partial B_0}M_{11}+\frac{\partial^2\dot{C}_1}{\partial B_0^2}(M_{10}M_{02}+M_{01}M_{11})+$$
$$+\frac{\partial\dot{C}_2}{\partial B_0}M_{02}+\frac{1}{2}\frac{\partial^3\dot{C}_1}{\partial A_0^2\,\partial B_0}M_{10}+\frac{\partial^3\dot{C}_1}{\partial A_0\,\partial B_0^2}M_{10}M_{01}+$$
$$+\frac{1}{2}\frac{\partial^3\dot{C}_1}{\partial B_0^3}M_{01}^2M_{10}+\frac{1}{2}\frac{\partial^2\dot{C}_2}{\partial A_0^2}+\frac{\partial^2\dot{C}_2}{\partial A_0\,\partial B_0}M_{01}+\frac{1}{2}\frac{\partial^2\dot{C}_2}{\partial B_0^2}M_{01}^2$$

Suppose that $K \neq 0$. From (2.14) it can be seen that there exists an expansion for v in the form of a series in powers of $\mu^{1/2}$

$$v=\sum_{n=1}^{\infty}A_{(n+2)/12}\mu^{n/2} \tag{2.18}$$

where $A_{(n+2)/2}$ are constant coefficients determined by the equations

$$\begin{aligned}&N_{02}A_{3/2}^2+K=0\\&2N_{02}A_{3/2}A_2+N_{21}A_{3/2}+2N_{12}aA_{3/2}+3N_{03}a^2A_{3/2}=0\\&2N_{02}A_{3/2}A_{5/2}+\ldots=0 \qquad \text{etc.}\end{aligned} \tag{2.19}$$

From the first equation of (2.19) we obtain $A_{3/2}$, and we find that $A_{3/2} \neq 0$ since $K \neq 0$.

The remaining equations are linear in the unknowns. A_2, $A_{5/2}$, ... whose coefficients are equal $2N_{02}A_{3/2}$. Since this quantity is different from zero, Equation (2.19) has a unique solution.

We note that the first equation of (2.19) yields two values of $A_{3/2}$

$$A_{3/2}^{(1)}=\sqrt{-KN_{02}^{-1}}, \qquad A_{3/2}^{(2)}=-\sqrt{-KN_{02}^{-1}}$$

If these roots are to be real it is necessary and sufficient that $N_{02}K < 0$.

From Equations (2.12), (2.5) and (2.18) we obtain

$$\beta_1^{(k)}=\mu a+A_{3/2}^{(k)}\mu^{3/2}+A_2^{(k)}\mu^2+\ldots$$
$$\beta_2^{(k)}=\mu B_1^{(k)}+B_{3/2}^{(k)}\mu^{3/2}+B_2^{(k)}\mu^2+\ldots$$

where

$$B_1^{(k)}=M_{10}+aM_{01}, \qquad B_{3/2}^{(k)}=M_{01}A_{3/2}^{(k)}$$
$$B_2^{(k)}=M_{20}+A_2^{(k)}M_{01}+M_{02}a^2+M_{11}a$$

Hence, if $K \neq 0$, $a_1 = a_2$ and $N_{02}K < 0$, then $\beta_1(\mu)$ and $\beta_2(\mu)$ will have two expansions with real coefficients in powers of $\mu^{1/2}$

c) Suppose $K = 0$. Then the first equation of (2.19) yields the equation $A_{3/2} = 0$. The remaining equations of the system will become indeterminate. However, when $K = 0$ Equation (2.14) becomes

$$v^2 N_{02} + \mu v (N_{21} + 2N_{12}a + 3N_{03}a^2) + \mu^2 L + \ldots = 0$$

Hence, one obtains an equation of the type (2.6) with $N_{10} = 0$, which can be investigated with the aid of the above described procedure and the use of a substitution of the type (2.12).

It is easy to see that the forms of the expansions will depend in this case on the multiplicity of the roots b_i of the equation

$$R = L + b (N_{21} + 2N_{12}a + 3N_{03}a^2) + b^2 N_{02} = 0$$

If this equation has single roots, there exist expansions $\beta_1(\mu)$ and $\beta_2(\mu)$ in integer powers of μ; if the equation has multiple roots, then one has to reconsider two cases, and so on. The analysis is analogous to the preceding one. The indicated transformations, no matter how often they are applied, always lead to an equation of the type (2.6) because $N_{02} \neq 0$.

3. Let us assume that Equations (1.5) are identities. Then all the derivatives of C_1 and C_1 with respect to A_0 and B_0 will be equal to zero. Dividing the terms of Equations (1.4) by μ, we obtain

$$\begin{aligned} C_2 + \frac{\partial C_2}{\partial A_0}\beta_1 + \frac{\partial C_2}{\partial B_0}\beta_2 + C_3\mu + \ldots = 0 \\ \dot{C}_2 + \frac{\partial \dot{C}_2}{\partial A_0}\beta_1 + \frac{\partial \dot{C}_2}{\partial B_0}\beta_2 + \dot{C}_3\mu + \ldots = 0 \end{aligned} \qquad (3.1)$$

If these equations are to be satisfied when $\mu \to 0$, it is necessary and sufficient that

$$C_2(2\pi) = 0, \qquad \dot{C}_2(2\pi) = 0 \qquad (3.2)$$

These equations serve in this case for the determination of the fundamental amplitudes, provided they do not reduce to identities. Equations (3.1) can be written in a form analogous to (1.4) which was studied in Section 2.

Obviously, if Equations (4.2) are identities then we shall have to use the equations

$$C_3(2\pi) = 0, \qquad \dot{C}_3(2\pi) = 0$$

for the determination of the amplitudes A_0 and B_0.

4. We shall next show how one can find in a practical manner the periodic solutions of Equation (1.1), whose existence under certain conditions was established in Section 2. It is easy to see from (1.2) and (1.3) that the form of the expansion of the solution $x(t, \beta_1, \beta_2, \mu)$ corresponds to the type of the expansions of $\beta_1(\mu)$ and $\beta_2(\mu)$ in terms of μ. Therefore, in the case of double roots of the equations of the fundamental amplitudes there exist real expansions of the periodic solutions of the system (1.1) in the form

$$x(t) = \sum_{n=1}^{\infty} x_{n/l}(t)\,\mu^{n/l} \qquad (4.1)$$

where $x_{n/l}(t)$ are periodic functions, and l is either one or two. There exist no expansions in terms of fractional powers of μ. Let us consider all the cases of Section (2).

1. The solution is sought in the form of a series in powers of $\mu^{1/2}$, namely,

$$x^{(k)}(t) = x_0(t) + x_{1/2}^{(k)}(t)\,\mu^{1/2} + x_1^{(k)}(t)\,\mu + x_{3/2}^{(k)}(t)\,\mu^{3/2} + \ldots \qquad (k = 1, 2)$$

where $x_i^{(k)}(t)$ $(i = 0, 1/2, 1, \ldots)$ are functions determined by means of (1.2) and (2.9):

$$x_{1/2}^{(k)}(t) = A_{1/2}^{(k)} \cos mt + \frac{B_{1/2}^{(k)}}{m} \sin mt$$

$$x_1^{(k)}(t) = A_1^{(k)} \cos mt + \frac{B_1^{(k)}}{m} \sin mt + C_1(t) \qquad \text{etc.}$$

2. If the roots of Equation (2.13) are simple, then the solution has the form (4.1) when $l = 1$.

$$x^{(k)}(t) = x_0(t) + x_1^{(k)}(t)\,\mu + x_2^{(k)}(t)\,\mu^2 + \ldots$$

where

$$x_1^{(k)}(t) = a_k \cos mt + \frac{1}{m}(M_{10} + M_{01}a_k) \sin mt + C_1(t)$$

$$x_2^{(k)}(t) = A_2 \cos mt + \frac{1}{m}(M_{20} + M_{01}A_2 + M_{02}a_k^2 + 2M_{02}a_kA_2 + M_{11}a_k) \sin mt +$$

$$+ C_2(t) + \frac{\partial C_1(t)}{\partial A_0} a_k + \frac{\partial C_1(t)}{\partial B_0}(M_{10} + M_{01}a_k) \qquad \text{etc.}$$

For multiple roots of Equation (2.13) when $K \neq 0$, we seek the solution in the form (4.1) with $l = 2$.

$$x^{(k)}(t) = x_0(t) + x_{1/2}^{(k)}(t)\,\mu^{1/2} + x_1^{(k)}(t)\,\mu + \ldots$$

where

$$x_{1/2}^{(k)}(t) = 0$$
$$x_1^{(k)}(t) = a\cos mt + \frac{1}{m}(M_{10} + aM_{01})\sin mt + C_1(t) \qquad \text{etc.}$$

There will exist no real expansions (4.1) if,

1) The condition (2.11) is not satisfied;

2) One of the following cases occurs; either the roots of Equation (2.13) are complex, or $a_1 = a_2$ and $N_{02}K > 0$.

In case of double roots of Equation (1.5) there exist unique expressions for the solutions in the form (4.1). In each case there exist two periodic solutions corresponding to the two multiple roots of the equations of the fundamental amplitudes. One can speak in this case of a bifurcation of the solution of the generated equation.

All the series (4.1) that have been obtained converge for small values of the parameter μ. In this work we have not considered the question of the stability of the obtained periodic solutions.

BIBLIOGRAPHY

1. Malkin, I.G., *Nekotorye zadachi teorii nelineinykh kolebanii (Some Problems of the Theory of Nonlinear Oscillations)*. Gostekhizdat, 1956.

2. Proskuriakov, A.P., Kolebaniia kvazilineinykh neavtonomnykh sistem s odnoi stepen'iu svobody vblizi rezonansa (Oscillations of quasi-linear nonautonomous systems with one degree of freedom near resonance). *PMM* Vol. 23, No. 4, 1959.

3. Erugin, N.P., *Neiavnye funktsii (Implicit Functions)*. Izd-vo LGU, 1956.

4. Goursat, E., *Kurs matematicheskogo analiza (Course in Mathematical Analysis)*, (Russian translation from the French). GITTL, 1936.

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